

Criticality in the $2 + 1$ -dimensional compact Higgs model and fractionalized insulators

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We use a novel method of computing the third moment M_3 of the action of the $2 + 1$ -dimensional compact Higgs model in the adjoint representation with $q = 2$ to extract correlation length and specific heat exponents ν and α without invoking hyperscaling. Finite-size scaling analysis of M_3 yields the ratios $(1 + \alpha)/\nu$ and $1/\nu$ separately. We find that α and ν vary along the critical line of the theory, which however exhibits a remarkable resilience of Z_2 criticality. We propose this novel universality class to be that of the quantum phase transition from a Mott-Hubbard insulator to a charge-fractionalized insulator in two spatial dimensions.

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Modelling of strongly correlated systems plays a central role in trying to understand unconventional metallic states in cuprate perovskites and other systems, which do not conform to the Landau Fermi-liquid paradigm [1]. One avenue of research attempting to establish a theory of non-Fermi liquids in more than one spatial dimension, focuses attention on effective gauge theories of matter fields representing the charge of doped Mott-Hubbard insulators, coupled to compact gauge fields emerging from strong constraints on the dynamics of the fermions [2, 3, 4, 5, 6]. Compact $U(1)$ gauge fields exhibit topological defects in the form of monopole configurations. It has been suggested that the unbinding of such monopoles may be relevant for spin-charge separation in strongly correlated systems [3, 6, 7] and for describing quantum antiferromagnets when fluctuations around the flux-phase are taken into account [8]. One often arrives at a description in terms of three-dimensional $d = 3$ compact QED ($cQED_3$). A formulation of charge-fractionalization in terms of a Z_2 lattice gauge theory coupled to matter fields, has also been put forth [9, 10, 11]. The above provides a link between important phenomena in condensed matter physics and deep issues in high-energy physics, such as confinement in QCD, with which $cQED_3$ shares two essential features, namely confinement and chiral symmetry breaking.

One lattice model arrived at in this context is the compact Higgs model defined by the partition function [3, 6, 12, 13, 14]

$$Z = \int_{-\pi}^{\pi} \left[\prod_{j=1}^N \frac{d\theta_j}{2\pi} \right] \int_{-\pi}^{\pi} \left[\prod_{j,\mu} \frac{dA_{j\mu}}{2\pi} \right] \exp[S]$$

$$S = \beta \sum_{j,\mu} [1 - \cos(\Phi_{\mu j})] + \kappa \sum_P [1 - \cos(\mathcal{A}_{\mu j})], \quad (1)$$

where N is the number of lattice sites, \sum_P runs over the plaquettes of the lattice, $\Phi_{\mu j} \equiv \Delta_{\mu}\theta_j - qA_{j\mu}$, and $\mathcal{A}_{\mu j} \equiv \varepsilon_{\mu\nu\lambda}\Delta_{\nu}A_{j\lambda}$. We use the variables ($x = 1/[\kappa + 1]$, $y = 1/[\beta + 1]$) in discussing the possible phases of this

model [12]. In Eq. (1), θ is the phase of a scalar matter field with unit norm representing holons, Δ_{μ} is a forward lattice difference operator in direction μ , while $A_{j\mu}$ is a fluctuating gauge field enforcing the onsite constraints from strong correlations in the problem.

When $q = 0$, the matter field decouples from the gauge field. The model has one critical point in the universality class of the $3DXY$ model, $y_c = 0.688$ ($y_c = 0.75$ in the Villain-approximation), while the pure gauge theory is permanently confined for all values of κ [15]. When $q = 1$, Eq. (1) is trivial on the line $x = 1, 0 < y < 1$, with no phase transition for any value of y . On the line $0 < x < 1, y = 1$ the matter field is absent and the theory is permanently confined [15]. The phase-structure for $q = 2, d = 3$ was briefly discussed in Ref. 13 and subsequently investigated numerically [16], the phase diagram is known, cf. Fig. 5 of Ref. 16. When $y \rightarrow 0$ there is an Ising transition at $x_c = 0.5678$, when $x \rightarrow 0$ there is a $3DXY$ critical point at $y_c = 0.688$. Moreover, a critical line $\beta_c(\kappa)$ connects these two critical limits. Above $\beta_c(\kappa)$ the system resides in a *deconfined-Higgs phase*, while below $\beta_c(\kappa)$ it resides in a *confined phase*. We also note that the model in Eq. (1) with $q = 2$ recently was proposed as an effective theory of a microscopic boson lattice model exhibiting charge-fractionalized phases [14]. The confined phase of Eq. (1) is interpreted as a Mott-Hubbard insulating phase, while the deconfined-Higgs phase is interpreted as a charge-fractionalized insulating phase [10, 11, 14].

No ordinary second order phase transition takes place in the case $d = 3, q = 1$ [13, 17]. However, two of us have recently shown [18] that when matter is coupled to a compact gauge-field in a *continuum theory* and treating the topological defects of the gauge-field in an analogous manner to that done in Ref. 15, the permanent confinement of the pure gauge theory is destroyed. A confinement-deconfinement transition may take place via a Kosterlitz-Thouless like unbinding of monopole configurations [18] in *three dimensions* due to the appearance of an anomalous scaling dimension of the gauge-field in-

duced by critical matter-field fluctuations [19]. The role of an anomalous scaling dimension has also been studied recently at finite temperature, in pure compact QED in $d = 3$ with no matter fields present [20]. In both Refs. 18, 20, the appearance of an anomalous scaling dimension is crucial. The authors of Ref. [20] recently also considered Eq. (1) with $q = 1$ numerically, [21], finding a recombination of monopoles into dipoles connected by matter strings, consistent with Ref. [18].

Given the relevance of the case $q = 2$ to current central issues in condensed matter physics [9, 10, 11, 14],

the universality class of the phase transition across the critical line for $q = 2$ warrants attention. We therefore compute the critical exponents α and ν . Our results i) *demonstrate* that the critical behavior found in the limits $\beta \rightarrow \infty$ (Z_2) and the limit $\kappa \rightarrow \infty$ ($U(1)$) are not isolated points, and ii) *on balance suggest* that Eq. (1) is a *fixed-line* with non-universal α and ν depending on (β, κ) rather than exhibiting a Z_2 - and a XY universality class separated by a multicritical point.

We express Eq. (1) as follows [6, 12]

$$Z = Z_0(\beta, \kappa) \sum_{\{Q_j\}} \sum_{\{J_{j\nu}\}} \delta_{\Delta_\nu J_{j\nu}, qQ_j} \exp \left[-4\pi^2 \beta \sum_{j,k} \left(J_{j\nu} J_{k\nu} + \frac{q^2}{m^2} Q_j Q_k \right) D(j-k, m^2) \right], \quad (2)$$

where δ is the Kronecker-delta, $D(j-k, m^2) = (-\Delta_\lambda^2 + m^2)^{-1} \delta_{jk}$, and $m^2 = q^2 \beta / \kappa$. $Z_0(\beta, \kappa)$ is the partition function for massive spin waves and will hereafter be omitted. Note the constraint $\Delta_\nu J_{j\nu} = qQ_j$ in the functional integral. Here Q_j is the monopole charge on the dual lattice site number j , while $J_{j\nu}$ are topological currents representing segments of either open-ended strings terminating on monopoles, or closed loops [12].

In the limit $\beta \rightarrow \infty$ at fixed κ , Eq. (2) takes the form

$$Z = \sum_{\{Q\}} \sum_{\{J_{j\nu}\}} \delta_{\Delta_\nu J_{j\nu}, qQ_j} \exp \left(-\frac{2\pi^2 \kappa}{q^2} \sum_j J_{j\nu}^2 \right). \quad (3)$$

This is the loop-gas representation of the global Z_q spin model in the Villain approximation [22]. From Eq. (3), it is seen that the cases $q = 1$ and $q \neq 1$ are fundamentally different. For $q = 1$, the summations over $\{Q_i\}$ may be performed to produce a unit factor at each dual lattice site, eliminating the constraint. Hence, $Z = \left(\vartheta_3(0, e^{-2\pi^2 \kappa}) \right)^N$ where ϑ_3 is an elliptic Jacobi function. No phase transition occurs at any value of κ for $q = 1$ when $\beta \rightarrow \infty$. For $q = 2$, a phase transition survives [13]. In the language of Eq. (3), this crucially depends on the presence of the constraint $\Delta_\nu J_{j\nu} = qQ_j$. For $q \neq 1$, summing over *all* values of $\{Q_j\}$ still provides a remnant constraint ensuring a theory sustaining a phase transition.

We compute the third moment M_3 of the action Eq. (3), $S = (2\pi^2 \kappa / q^2) \sum_j J_{j\nu}^2 \equiv (2\pi^2 \kappa / q^2) H$, with M_n given by

$$M_n = \langle (H - \langle H \rangle)^n \rangle. \quad (4)$$

Using finite-size scaling (FSS) at the critical point, the peaks in M_n scale with system size L as $L^{(n-2+\alpha)/\nu}$. The

width between the peaks in M_3 scales as $L^{-1/\nu}$, see Fig. 1. Thus *both* α and ν are found from computing *one* quantity without using hyperscaling. Also, FSS of M_3 provides superior quality scaling compared to M_2 (specific heat), which unfortunately often is marred by significant confluent singularities. We have used multihistogram reweighting [23] of rawdata and jackknifing for error estimation in M_3 to perform FSS of the peak height and width between peaks.

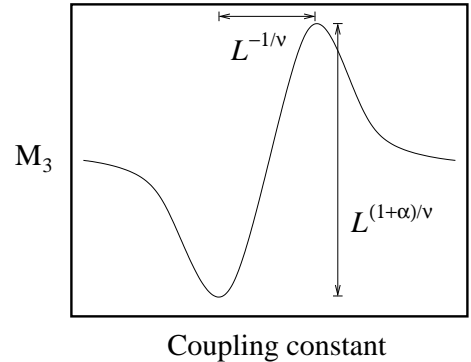


FIG. 1: Generic third moment of action, M_3 , showing how FSS is used to extract α and ν .

Before computing M_3 of Eq. (3), and Eq. (1), we perform benchmark Monte-Carlo simulations (MCS) on three well-known models. In Fig. 2 a) we show FSS results for the height of the peaks in M_3 , defined analogously to Eq. (4), for the 3D Ising- and 3DXY-models for $L = 8, 12, 16, 20, 32, 40, 64$, with standard Metropolis updating. These are limiting cases of Eq. (1) (see below). Moreover (see below) the 3D Ising spin model is dual to the 3D Ising gauge theory (IGT) [24], and we have thus also computed M_3 for IGT. Any action must have α and ν identical with those of its dual counterpart,

since α and ν can be obtained directly from scaling of the free energy, and are independent of the degrees of freedom one chooses to describe the system in terms of. Our simulations bear this out with precision, cf. Fig. 2 a), providing a nontrivial quality check on them.

The system sizes we have used for the MCS on Eq. (3) are L^3 , with $L = 8, 12, 16, 24, 32, 48, 64$, results are shown in Fig. 2 b) (\square -symbols). The allowed MC moves using Eq. (3) are i) insertions of elementary loops made of vortex segments $J_{j\nu} = \pm 1$ and ii) insertions of open-ended vortex segments $J_{j\nu} = \pm q$ satisfying the constraint in Eq. (3). Up to $4 \cdot 10^6$ sweeps over the lattice have been used, with periodic boundary conditions in all directions.

From Eq. 1, the limit $\beta \rightarrow \infty$, κ fixed leads to the constraint $\Delta_\mu \theta_j - q A_{j\mu} = 2\pi l_{j\mu}$ where $l_{j\mu}$ is integer valued. Substituting this into the gauge-field term in Eq. (1), we find

$$Z = \prod_{j=1}^N \sum_{l_{j,\mu}=-\infty}^{\infty} \exp \left[\kappa \sum_P \left(1 - \cos \left(\frac{2\pi \mathcal{L}_{\mu j}}{q} \right) \right) \right], \quad (5)$$

where $\mathcal{L}_{\mu j} = \varepsilon_{\mu\nu\lambda} \Delta_\nu l_{j\lambda} \in \mathbb{Z}$. For $q = 1$ the model is again seen to be trivial. Since Eqs. (3) and (5) are dual, and Eq. (3) is a loop-gas representation of the global Z_q theory while Eq. (5) is the Z_q lattice gauge theory, it follows that the global and local Z_q theories are dual in $d = 3$ [25]. Hence, the model Eq. (1) in the limit $\beta \rightarrow \infty$, κ fixed, should have a ratio $(1 + \alpha)/\nu$ consistent with the Z_q spin model universality class, if the transition is continuous.

For $q = 2$, we have performed large-scale MCS and FSS analysis of M_3 of the action S in Eq. 1, written as $S = \beta H_\Phi + \kappa H_A$, with $H_\lambda = \sum [1 - \cos(\lambda_{\mu j})]$, cf. Eq. (1). A critical line $\beta_c(\kappa)$ separates a confined ($\beta < \beta_c$) and a Higgs-deconfined ($\beta > \beta_c$) state [16]. We have used $L = 8, 12, 16, 20, 24, 32, 40$, and up to $9 \cdot 10^6$ sweeps over the lattice with periodic boundary conditions in all directions. The critical line is crossed along the trajectory $\beta(\kappa) = \beta_c + a(\kappa - \kappa_c)$, where (β_c, κ_c) is a point on the critical line. For the points at which $(1 + \alpha)/\nu$ has extrema, we use $a = (-1, 1, \infty)$ to check that values for α and ν are not artifacts of how the critical line is crossed.

In Fig. 2 b) we show scaling plots of the peaks in M_3 for Eq. (1) with $q = 2$ for various values of κ/β on the critical line, Fig. 2 c) shows corresponding scaling plots of the width between the peaks. From the finite-size scaling of the features in M_3 , Fig. 1, we extract the combination $(1 + \alpha)/\nu$ as well as the exponent $1/\nu$ (and hence α) along the critical line, the results are shown in Fig. 3. In Fig. 3 c), we also give values of α obtained directly from M_3 as well as using $(1 + \alpha)/\nu$ together with hyperscaling $\alpha = 2 - d\nu$. We have checked that the extrema in $(1 + \alpha)/\nu$ are not changed when the critical line is crossed in three very different directions, using $a = -1$, $a = 1$ and $a = \infty$.

The results seem to rule out that Z_2 - and XY -critical

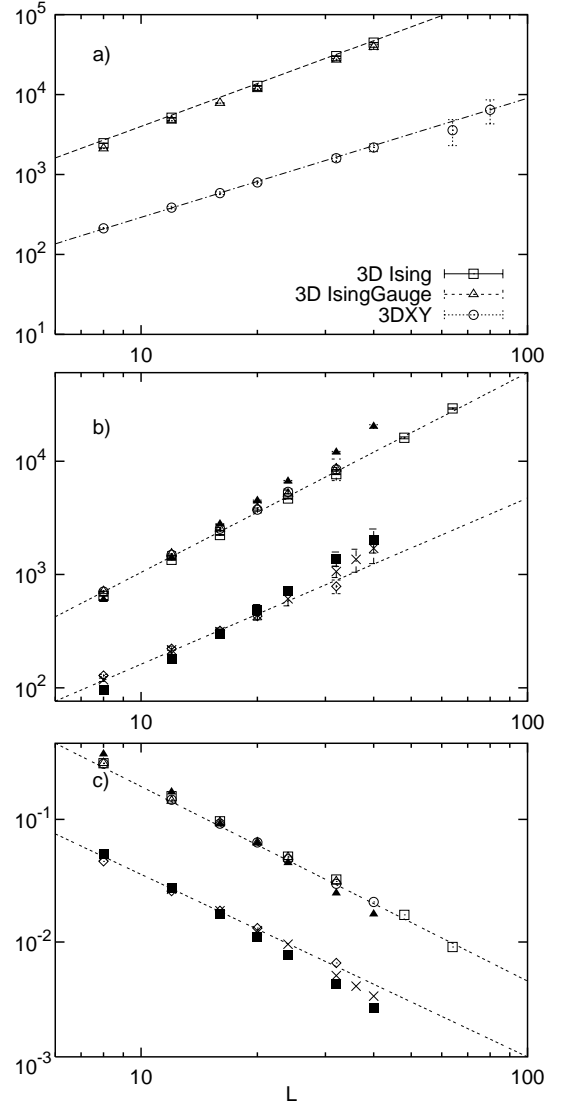


FIG. 2: Log-log scaling plots of peak-heights in M_3 . a) 3D Ising-, Ising gauge-, and XY-models. b) Eq. (1) with $q = 2$, $\kappa/\beta = 0.303$ (\circ), $\kappa/\beta = 1.00$ (\triangle), $\kappa/\beta = 2.88$ (\blacktriangle), $\kappa/\beta = 3.47$ (\blacksquare), $\kappa/\beta = 4.05$ (\times), $\kappa/\beta = 5.14$ (\diamond). Also shown are results for Eq. (3) (\square). c) Scaling plots of width between peaks in M_3 for Eq. (3), and Eq. (1) for $q = 2$, legends as in b).

behaviors are isolated points at the extreme ends of the critical line. However, from Fig. 3, it is feasible to suggest two types of universality, Z_2 and XY , separated at a multicritical point. We believe this to be ruled out by the strong deviation in $(1 + \alpha)/\nu$ from Z_2 - and XY -values at intermediate κ/β , which are insensitive to a . On balance, we thus conclude that the model Eq. (1) defines a *fixed-line theory*, rather than exhibiting two scaling regimes separated by a multicritical point. However, the Z_2 character of the confinement-deconfinement transition persists to surprisingly large values of κ/β on the critical line, cf. Fig. 5 of Ref. 16. Fixed-line theories

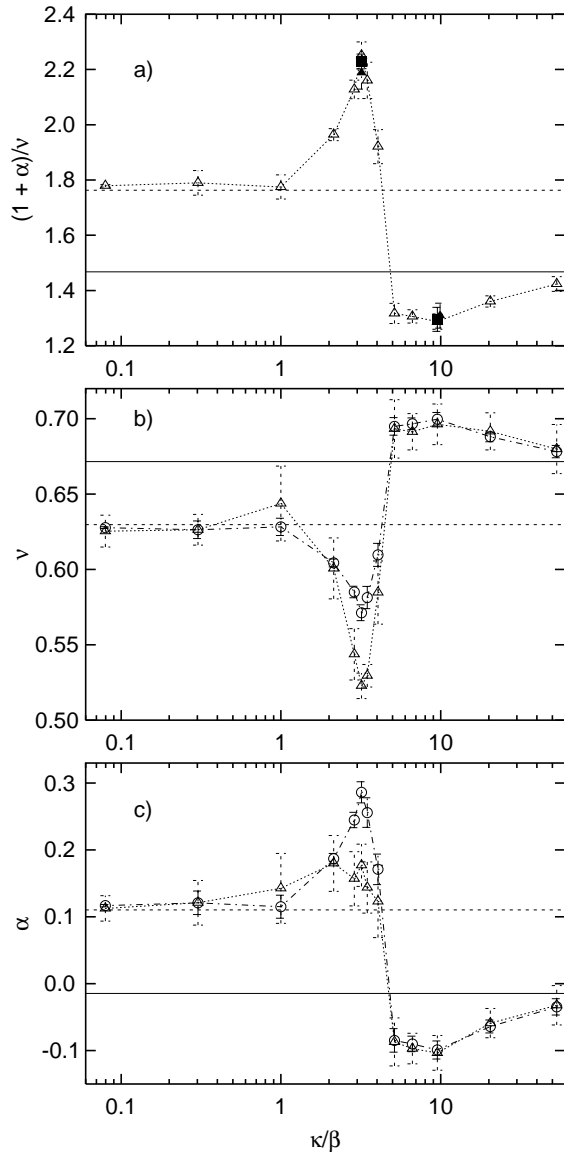


FIG. 3: a) $\Upsilon \equiv (1 + \alpha)/\nu$ from FSS finite-size of M_3 for Eq. (1) for $q = 2$. b) Same for the exponent ν , computed directly from M_3 (\triangle) and using results for $(1 + \alpha)/\nu$ with hyperscaling (\circ). c) α as computed directly from M_3 (\triangle) and using hyperscaling (\circ). The maximum and minimum in a) have been obtained by crossing the critical line along the trajectory $\beta(\kappa) = \beta_c + a(\kappa - \kappa_c)$ with $a = \infty$ (\triangle), $a = 1$ (\blacksquare), and $a = -1$ (\blacktriangle) using $\beta_c = 0.665$, $\kappa_c = 2.125$ (max.), and $\beta_c = 0.525$, $\kappa_c = 5.0$ (min.). At the maximum, $\Upsilon(a = \infty) = 2.25 \pm 0.05$, $\Upsilon(a = 1) = 2.23 \pm 0.03$, $\Upsilon(a = -1) = 2.19 \pm 0.06$. At the minimum, $\Upsilon(a = \infty) = 1.30 \pm 0.04$, $\Upsilon(a = 1) = 1.31 \pm 0.05$, $\Upsilon(a = -1) = 1.29 \pm 0.03$. Dotted horizontal lines indicate Z_2 -values, solid horizontal lines indicate $U(1)$ -values.

in $2 + 1$ dimensions are known [26], and non-universal exponents imply the existence of marginal operators in Eq. (1), yet to be identified.

Recently, Eq. (1) with $q = 2$ was proposed as an effective theory for a microscopic model exhibiting a quan-

tum phase transition from a Mott Hubbard insulator to a charge-fractionalized insulator in two spatial dimension [14]. We thus propose that the zero temperature quantum phase transition from a Mott-Hubbard insulator to a charge-fractionalized insulator [11, 14] is characterized by a *fixed-line theory* as given in Fig. 3, but with remarkable Z_2 resilience.

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